

SDEs with constraints driven by processes with bounded p -variation

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Abstract

We study the existence, uniqueness and approximation of solutions of stochastic differential equations with constraints driven by processes with bounded p -variation. Our main tool are new estimates showing Lipschitz continuity of the deterministic Skorokhod problem in p -variation norm. Applications to fractional SDEs with constraints are given.

Key Words: Skorokhod problem, p -variation, integral equations, stochastic differential equations with constraints, reflecting boundary condition.

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1 Introduction

In the present paper we study the problems of existence, uniqueness and approximation of solutions of finite-dimensional stochastic differential equations (SDEs) with constraints driven by general processes with bounded p -variation, $p \geq 1$. More precisely, let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $g : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ be measurable functions, A be a one-dimensional process with locally bounded variation and Z be a d -dimensional process with locally bounded p -variation. We consider SDEs with reflecting boundary condition of the form

$$X_t = X_0 + \int_0^t f(X_{s-}) dA_s + \int_0^t g(X_{s-}) dZ_s + K_t, \quad t \in \mathbb{R}^+. \quad (1.1)$$

By a solution to (1.1) we mean a pair (X, K) consisting of a process X living over a given d -dimensional barrier process L and a d -dimensional process K , called regulator term, whose each component K^i is nondecreasing and increases only when X^i is living on L^i (for details see Section 3). Equation (1.1) is called the Skorokhod SDE in analogy with the case $L = 0$ first discussed by Skorokhod [26] for a standard Brownian motion in place of Z and $A_t = t$, $t \in \mathbb{R}^+$. Next, many attempts have been made to extend Skorokhod's results to larger class of domains or larger class of driving processes (see, e.g., [3, 9, 19, 30, 33]). This kind of equations have many applications, for instance in queueing systems, seismic reliability analysis and finance (see, e.g., [1, 10, 16, 25]). In recent papers by Besalu and Rovira [2] and

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Ferrante and Rovira [13] SDE with non-negativity constraints driven by fractional Brownian motion B^H with Hurst index $H > 1/2$ and $A_t = t$, $t \in \mathbb{R}^+$, is studied. This equation is a particular case of (1.1) because B^H has locally bounded p -variation for $p > 1/H$. In the main theorem of [13] the existence of a solution is proved under the assumption that the coefficients f, g are Lipschitz continuous. The proof is based on a quite natural in the context of SDEs driven by B^H technics based on λ -Hölder norms. Unfortunately, in [13] it is only shown that the solution is unique for some small time interval. To our knowledge, global uniqueness for fractional SDEs with constraints is still an open problem. In contrast to [13], in our paper we use p -variation norm (for the theory of functions of p -variation and its various applications see, e.g., [7, 8]).

In our paper we consider two conditions: continuity and linear growth of f and Hölder continuity of g (condition (H1)) and local Lipschitz continuity of f and local Hölder continuity of the derivative of each component $g_{i,j}$ $i, j = 1, \dots, d$ (condition (H2)) (see Section 3). We show that under (H1) and (H2) there exists a unique (globally in time) solution to (1.1), which can be approximated by some natural approximation schemes.

The paper is organised as follows.

In Section 2 we consider the deterministic Skorokhod problem $x = y + k$ associated with $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and time dependent lower barrier $l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ with $l_0 \leq y_0$. We show that the mapping $(y, l) \mapsto (x, k)$ is Lipschitz continuous in p -variation norm. In fact, we show that if (x, k) , is a solution associated with $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and barrier $l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and (x', k') , is a solution associated with $y' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and barrier $l' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ then for any $T \in \mathbb{R}^+$,

$$\bar{V}_p(x - x')_T \leq (d + 1)\bar{V}_p(y - y')_T + d\bar{V}_p(l - l')_T \quad (1.2)$$

and

$$\bar{V}_p(k - k')_T \leq d\bar{V}_p(y - y')_T + d\bar{V}_p(l - l')_T. \quad (1.3)$$

It is worth noting here that in [13, Remark 3.6] it is observed that $(y, l) \mapsto (x, k)$ is not Lipschitz continuous in the λ -Hölder norm and for that reason in [13] the authors were not able to obtain global uniqueness.

In Section 3 we consider a deterministic counterpart to (1.1). We prove that under (H1) the deterministic equation has a solution. If moreover (H2) is satisfied that it is unique. Then we show convergence of some natural approximation schemes for a deterministic equation of the form (1.1). In the proofs of convergence we use the Skorokhod topology J_1 and general methods of approximations of stochastic integrals and solutions of SDEs developed in [15, 21, 27, 28]. For the convenience of the reader we prove in Appendix a general tightness criterion and a functional limit theorem for sequences of integrals with respect to càdlàg functions with bounded p -variation.

In Section 4 we apply our deterministic results to obtain the existence, uniqueness and approximation of solutions to SDEs of the form (1.1). In particular, we show that if f, g satisfy (H1) and (H2) then (1.1) has a unique strong solution (X, K) . Moreover, we show convergence to (X, K) of some easy to implement approximations (X^n, K^n) constructed in analogy with the classical Euler scheme. To illustrate how our results work in practice, at the end of the paper we consider fractional SDEs with constraints of the form

$$X_t = X_0 + \int_0^t f(X_{s-}) da_s + \int_0^t g(X_{s-}) dZ_s^H + K_t, \quad t \in \mathbb{R}^+. \quad (1.4)$$

Here $a : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function with locally bounded variation and $Z^{H,i} = \int_0^\cdot \sigma_s^i dB_s^{H,i}$, $t \in \mathbb{R}^+$, where $B^{H,1}, \dots, B^{H,d}$ are independent fractional Brownian motions and $\sigma^i : \mathbb{R}^+ \rightarrow \mathbb{R}$ are such that $\|\sigma^i\|_{\mathbb{L}_{[0,T]}^{1/H}} := (\int_0^T |\sigma_s^i|^{1/H} ds)^H < \infty$, $T > 0$, $i = 1, \dots, d$. Under the last assumption Z^H is a centered Gaussian process with continuous trajectories such that $P(V_p(Z^H)_T < \infty) = 1$, $p > 1/H$, $T \in \mathbb{R}^+$ (see Section 4), so (1.4) is a particular case of (1.1). Of course, (1.4) generalizes classical fractional SDEs driven by B^H .

In the sequel we will use the following notation. \mathbb{M}^d is the space of $d \times d$ real matrices A , with the matrix norm $\|A\| = \sup\{|Au|; u \in \mathbb{R}^d, |u| = 1\}$, where $|\cdot|$ denotes the usual Euclidean norm in \mathbb{R}^d , $\mathbb{R}^+ = [0, \infty)$. $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of càdlàg mappings $x : \mathbb{R}^+ \rightarrow \mathbb{R}^d$, i.e. mappings which are right continuous and admit left-hands limits equipped with the Skorokhod topology J_1 . For $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $t > 0$ we denote $x_{t-} = \lim_{s \uparrow t} x_s$ and $v_p(x)_{[a,b]} = \sup_\pi \sum_{i=1}^n |x_{t_i} - x_{t_{i-1}}|^p < \infty$, where the supremum is taken over all subdivisions $\pi = \{a = t_0 < \dots < t_n = b\}$ of $[a, b]$. $V_p(x)_{[a,b]} = (v_p(x)_{[a,b]})^{1/p}$ and $\bar{V}_p(x)_{[a,b]} = V_p(x)_{[a,b]} + |x_a|$ is the usual variation norm. For simplicity of notation we write $v_p(x)_T = v_p(x)_{[0,T]}$, $V_p(x)_T = V_p(x)_{[0,T]}$ and $\bar{V}_p(x)_T = \bar{V}_p(x)_{[0,T]}$. If $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$ then in the definition of p -variation v_p we use the matrix norm $\|\cdot\|$ in place of the Euclidean norm. We write $x \leq x'$, $x, x' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ if $x_t^i \leq x_t'^i$, $t \in \mathbb{R}^+$, $i = 1, \dots, d$. Every process Y appearing in the sequel is assumed to have càdlàg trajectories.

2 Lipschitz continuity of the solution of the Skorokhod problem in p -variation norm

Let $y, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $l_0 \leq y_0$. We recall that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is called a solution of the Skorokhod problem associated with y and lower barrier l ($(x, k) = SP_l(y)$ for short) if

- (i) $x_t = y_t + k_t \geq l_t$, $t \in \mathbb{R}^+$,
- (ii) $k_0 = 0$, $k = (k^1, \dots, k^d)$, where k^i are nondecreasing functions such that for every $t \in \mathbb{R}^+$,

$$\int_0^t (x_s^i - l_s^i) dk_s^i = 0, \quad i = 1, \dots, d.$$

The Lipschitz continuity of the mapping $(y, l) \mapsto (x, k)$ in the supremum norm is well known. More precisely, let $(x, k) = SP_l(y)$, $(x', k') = SP_{l'}(y')$. Since $k_t = \sup_{s \leq t} (y_s - l_s)^-$ and $k'_t = \sup_{s \leq t} (y'_s - l'_s)^-$, for any $T \in \mathbb{R}^+$ we have

$$\sup_{t \leq T} |x_t - x'_t| \leq 2 \sup_{t \leq T} |y_t - y'_t| + \sup_{t \leq T} |l_t - l'_t| \quad (2.5)$$

and

$$\sup_{t \leq T} |k_t - k'_t| \leq \sup_{t \leq T} |y_t - y'_t| + \sup_{t \leq T} |l_t - l'_t|. \quad (2.6)$$

On the other hand, it was observed in Ferrante and Rovira [13] that above property does not hold in λ -Hölder norm. We will show that the Lipschitz continuity of the mapping $(y, l) \mapsto (x, k)$ holds in the variation norm. A key step in proving it is the following estimate.

Theorem 2.1 For any $y^1, y^2 \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ and $T \in \mathbb{R}^+$,

$$v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_T \leq v_p(y^1 - y^2)_T.$$

PROOF. It is clear that without loss of generality we may and will assume that

$$v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_T > 0. \quad (2.7)$$

Step 1. We assume additionally that y^1, y^2 are step functions of the form

$$y_t^j = y_{j,i}, \quad t \in [t_{i-1}, t_i), \quad i = 1, \dots, n-1$$

and $y_t^j = y_{j,n}$, $t \in [t_{n-1}, t_n = T]$, $j = 1, 2$, for some partition $0 = t_0 < t_1 < \dots < t_n = T$ of the interval $[0, T]$.

Set $Y_k^j = \max_{1 \leq i \leq k} y_{j,i}$, $k = 1, \dots, n$, $j = 1, 2$. By (2.7) it is clear that there exists k such that $Y_k^1 > Y_{k-1}^1$ or $Y_k^2 > Y_{k-1}^2$. Without loss of generality we will assume that for any $k = 2, \dots, n$,

$$Y_k^1 > Y_{k-1}^1 \quad \text{or} \quad Y_k^2 > Y_{k-1}^2. \quad (2.8)$$

Indeed, if (2.8) does not hold then we set $u_0 = 0$,

$$u_k = \inf\{i > u_{k-1}; Y_i^1 > Y_{i-1}^1 \text{ or } Y_i^2 > Y_{i-1}^2\} \wedge n, \quad k = 1, \dots, n$$

and $\tilde{n} = \inf\{k; u_k = n\}$, $\tilde{y}_t^j = y_{j,u_k}$, $t \in [t_{u_{k-1}}, t_{u_k})$ for $k = 1, \dots, \tilde{n} - 1$, $\tilde{y}_t^j = y_{j,\tilde{n}}$ for $t \in [t_{u_{\tilde{n}-1}}, t_{u_{\tilde{n}}} = T]$, $j = 1, 2$. Then (2.8) holds true for the functions \tilde{y}^1, \tilde{y}^2 and $v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_T = v_p(\sup_{s \leq \cdot} \tilde{y}_s^1 - \sup_{s \leq \cdot} \tilde{y}_s^2)_T$, $v_p(\tilde{y}^1 - \tilde{y}^2)_T \leq v_p(y^1 - y^2)_T$.

It is clear that there exist numbers $0 = i_0 < i_1 < \dots < i_m = n$ such that

$$v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_T = \sum_{k=1}^m |(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \quad (2.9)$$

and

$$(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2) \neq 0 \quad (2.10)$$

for $k = 1, \dots, m$. In particular, this implies that if $m \geq 2$ then for $k = 2, \dots, m$ we have

$$((Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2))((Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)) < 0. \quad (2.11)$$

Indeed, if (2.11) is not satisfied then by (2.10),

$$\begin{aligned} & |(Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2)|^p + |(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \\ & < |(Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2) + (Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \\ & = |(Y_{i_k}^1 - Y_{i_{k-2}}^1) - (Y_{i_k}^2 - Y_{i_{k-2}}^2)|^p, \end{aligned}$$

which contradicts (2.9). Set $l_k^j = \max\{i \leq i_k : y_i^j = Y_{i_k}^j\}$, $j = 1, 2$, and $l_k^\wedge = \min\{l_k^1, l_k^2\}$, $l_k^\vee = \max\{l_k^1, l_k^2\}$, $k = 1, \dots, m$. Then

$$y_{1, l_k^1} = Y_{l_k^1}^1 = Y_{l_k^1+1}^1 = \dots = Y_{i_k}^1 \quad \text{and} \quad y_{2, l_k^2} = Y_{l_k^2}^2 = Y_{l_k^2+1}^2 = \dots = Y_{i_k}^2. \quad (2.12)$$

We claim that for any $k = 1, \dots, m$,

$$i_{k-1} \leq l_k^\wedge \leq l_k^\vee \leq i_k. \quad (2.13)$$

The last two inequalities are obvious. Moreover, $0 = i_0 \leq l_1^\wedge$. Assume that there exists $2 \leq k \leq m$ such that $i_{k-1} > l_k^\wedge$. In what follows we will only consider the case $l_k^\wedge = l_k^1$ (the case $l_k^\wedge = l_k^2$ can be handled in much the same way). We have $i_{k-2} < l_k^\wedge$, because if $i_{k-2} \geq l_k^\wedge$ then by (2.12),

$$\begin{aligned} & |(Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2)|^p + |(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \\ &= |(Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2)|^p + |(Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \\ &< |(Y_{i_k}^2 - Y_{i_{k-2}}^2)|^p \\ &= |(Y_{i_k}^1 - Y_{i_{k-2}}^1) - (Y_{i_k}^2 - Y_{i_{k-2}}^2)|^p, \end{aligned}$$

which contradicts (2.9). From the inequality $i_{k-2} < l_k^\wedge$ and (2.8) it follows that

$$Y_{i_{k-2}}^1 \leq Y_{l_k^1}^1 = Y_{l_k^1+1}^1 = \dots = Y_{i_{k-1}}^1 = \dots = Y_{i_k}^1 \quad (2.14)$$

and

$$Y_{i_{k-2}}^2 \leq Y_{l_k^2}^2 < Y_{l_k^2+1}^2 < \dots < Y_{i_{k-1}}^2 < \dots < Y_{i_k}^2. \quad (2.15)$$

Since

$$(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2) = -(Y_{i_k}^2 - Y_{i_{k-1}}^2) < 0,$$

(2.11), (2.14) and (2.15) imply that

$$\begin{aligned} 0 < (Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2) &= (Y_{l_k^1}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2) \\ &< (Y_{l_k^1}^1 - Y_{i_{k-2}}^1) - (Y_{l_k^1}^2 - Y_{i_{k-2}}^2), \end{aligned}$$

hence that

$$|(Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2)|^p < |(Y_{l_k^1}^1 - Y_{i_{k-2}}^1) - (Y_{l_k^1}^2 - Y_{i_{k-2}}^2)|^p. \quad (2.16)$$

Similarly,

$$\begin{aligned} 0 < -((Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)) &= (Y_{i_k}^2 - Y_{i_{k-1}}^2) \\ &< (Y_{i_k}^2 - Y_{l_k^1}^2) \\ &= (Y_{i_k}^1 - Y_{l_k^1}^1) - (Y_{i_k}^2 - Y_{l_k^1}^2), \end{aligned}$$

which implies that

$$|(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p < |(Y_{i_k}^1 - Y_{l_k^1}^1) - (Y_{i_k}^2 - Y_{l_k^1}^2)|^p. \quad (2.17)$$

Combining (2.16) with (2.17) we obtain

$$\begin{aligned} & |(Y_{i_{k-1}}^1 - Y_{i_{k-2}}^1) - (Y_{i_{k-1}}^2 - Y_{i_{k-2}}^2)|^p + |(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \\ &< |(Y_{l_k^1}^1 - Y_{i_{k-2}}^1) - (Y_{l_k^1}^2 - Y_{i_{k-2}}^2)|^p + |(Y_{i_k}^1 - Y_{l_k^1}^1) - (Y_{i_k}^2 - Y_{l_k^1}^2)|^p. \end{aligned}$$

which contradicts (2.9) and completes the proof of (2.13). It is clear that for any k we have $y_{1,l_k^1} \geq y_{1,l_k^2}$ and $y_{2,l_k^2} \geq y_{2,l_k^1}$. Consequently, in the case $Y_{i_k}^1 - Y_{i_{k-1}}^1 > Y_{i_k}^2 - Y_{i_{k-1}}^2$ we have

$$\begin{aligned} 0 &< (Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2) = (y_{1,l_k^1} - y_{1,l_{k-1}^1}) - (y_{2,l_k^2} - y_{2,l_{k-1}^2}) \\ &\leq (y_{1,l_k^1} - y_{1,l_{k-1}^2}) - (y_{2,l_k^1} - y_{2,l_{k-1}^2}). \end{aligned}$$

Hence

$$|(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \leq |(y_{1,l_k^1} - y_{1,l_{k-1}^2}) - (y_{2,l_k^1} - y_{2,l_{k-1}^2})|^p. \quad (2.18)$$

Similarly one can check that if $Y_{i_k}^1 - Y_{i_{k-1}}^1 < Y_{i_k}^2 - Y_{i_{k-1}}^2$ then

$$|(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \leq |(y_{1,l_k^2} - y_{1,l_{k-1}^1}) - (y_{2,l_k^2} - y_{2,l_{k-1}^1})|^p. \quad (2.19)$$

By (2.18) and (2.19),

$$\sum_{k=1}^m |(Y_{i_k}^1 - Y_{i_{k-1}}^1) - (Y_{i_k}^2 - Y_{i_{k-1}}^2)|^p \leq \sum_{k=1}^m |(y_{1,l_k} - y_{1,l_{k-1}}) - (y_{2,l_k} - y_{2,l_{k-1}})|^p,$$

where $l_k = l_k^1$ or $l_k = l_k^2$. Moreover, by (2.13), $i_{k-1} \leq l_k \leq i_k$ for $k = 1, \dots, m$. Hence

$$v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_T \leq \sum_{k=1}^m |(y_{l_k}^1 - y_{l_{k-1}}^1) - (y_{l_k}^2 - y_{l_{k-1}}^2)|^p$$

for some partition $0 = t_{l_0} < t_{l_1} < \dots < t_{l_m} \leq T$, which proves the theorem under our additional assumption.

Step 2. The general case.

Let $\{y^{1,n}\}$ and $\{y^{2,n}\}$ be sequences of discretizations of y^1 and y^2 , respectively, i.e. $y_t^{1,n} = y_{k/n}^1$, $y_t^{2,n} = y_{k/n}^2$, $t \in [k/n, (k+1)/n)$, $k \in \mathbb{N} \cup \{0\}$. By Step 1, for any $n \in \mathbb{N}$ and $T \in \mathbb{R}^+$ we have

$$v_p(\sup_{s \leq \cdot} y_s^{1,n} - \sup_{s \leq \cdot} y_s^{2,n})_T \leq v_p(y^{1,n} - y^{2,n})_T.$$

Clearly, $v_p(y^{1,n} - y^{2,n})_T \leq v_p(y^1 - y^2)_T$, $n \in \mathbb{N}$, $T \in \mathbb{R}^+$. By using e.g. [11, Chapter 3, Proposition 6.5] one can check that

$$y^{1,n} \longrightarrow y^1 \text{ and } y^{2,n} \longrightarrow y^2 \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}).$$

Hence and by [14, Chapter VI. Proposition 2.2]

$$(y^{1,n}, y^{2,n}) \longrightarrow (y^1, y^2) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^2),$$

which together with [14, Chapter VI. Proposition 2.4] implies that

$$\sup_{s \leq \cdot} y_s^{1,n} - \sup_{s \leq \cdot} y_s^{2,n} \longrightarrow \sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2 \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}).$$

Therefore for any T such that $\Delta y_T^1 = \Delta y_T^2 = 0$,

$$\begin{aligned} v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_T &\leq \liminf_{n \rightarrow \infty} v_p(\sup_{s \leq \cdot} y_s^{1,n} - \sup_{s \leq \cdot} y_s^{2,n})_T \\ &\leq \sup_n v_p(y^{1,n} - y^{2,n})_T \leq v_p(y^1 - y^2)_T. \end{aligned}$$

If $\Delta y_T^1 \neq 0$ or $\Delta y_T^2 \neq 0$ then there exists a sequence $\{T_k\}$ such that $T_k \downarrow T$ and $\Delta y_{T_k}^1 = \Delta y_{T_k}^2 = 0$, $k \in \mathbb{N}$. Then $v_p(\sup_{s \leq \cdot} y_s^1 - \sup_{s \leq \cdot} y_s^2)_{T_k} \leq v_p(y^1 - y^2)_{T_k}$, $k \in \mathbb{N}$, so letting $k \rightarrow \infty$ we obtain the desired result. \square

Theorem 2.2 *Assume $l, y, l', y' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ are such that $l_0 \leq y_0$ and $l'_0 \leq y'_0$. Let $(x, k) = SP_l(y)$ and $(x', k') = SP_{l'}(y')$. Then for any $T \in \mathbb{R}^+$,*

$$V_p(x - x')_T \leq (d+1)V_p(y - y')_T + d|y_0 - y'_0| + dV_p(l - l')_T + d|l_0 - l'_0|$$

and

$$V_p(k - k')_T \leq dV_p(y - y')_T + d|y_0 - y'_0| + dV_p(l - l')_T + d|l_0 - l'_0|.$$

PROOF. Observe that $k_t = \sup_{s \leq t} (y_s - l_s)^- = \sup_{s \leq 1+t} \bar{y}_s$, where $\bar{y}_s = 0$ for $s \in [0, 1)$ and $\bar{y}_s = l_{s-1} - y_{s-1}$ for $s \geq 1$. Similarly, $k'_t = \sup_{s \leq 1+t} \bar{y}'_s$, where $\bar{y}'_s = 0$ for $s \in [0, 1)$ and $\bar{y}'_s = l'_{s-1} - y'_{s-1}$ for $s \geq 1$. By Theorem 2.1,

$$\begin{aligned} V_p(k - k')_T &= V_p(\sup_{s \leq \cdot} \bar{y}_s - \sup_{s \leq \cdot} \bar{y}'_s)_{T+1} \\ &\leq d^{(p-1)/p} \left(\sum_{i=1}^d v_p(\sup_{s \leq \cdot} \bar{y}_s^i - \sup_{s \leq \cdot} \bar{y}'_s^i)_{T+1} \right)^{1/p} \\ &\leq d^{(p-1)/p} \left(\sum_{i=1}^d v_p(\bar{y}^i - \bar{y}'^i)_{T+1} \right)^{1/p} \\ &\leq d \max_i V_p(\bar{y}^i - \bar{y}'^i)_{T+1} \leq dV_p(\bar{y} - \bar{y}')_{[0, T+1]}. \end{aligned}$$

Since

$$\begin{aligned} V_p(\bar{y} - \bar{y}')_{T+1} &\leq V_p(\bar{y} - \bar{y}')_{[0, 1]} + V_p(\bar{y} - \bar{y}')_{[1, T+1]} \\ &= |(y_0 - y'_0) - (l_0 - l'_0)| + V_p((y - y') - (l - l'))_T \\ &\leq V_p(y - y')_T + |y_0 - y'_0| + V_p(l - l')_T + |l_0 - l'_0| \end{aligned}$$

and

$$V_p(x - x')_T \leq V_p(y - y')_T + V_p(k - k')_T,$$

the proof is complete. \square

Corollary 2.3 *Under the assumptions of Theorem 2.2, for every $T \in \mathbb{R}^+$ the estimates (1.2), (1.3) hold true.*

PROOF. It suffices to observe that $x_0 = y_0$, $x'_0 = y'_0$ and $k_0 = k'_0 = 0$. \square

Remark 2.4 (a) *The case $p = d = 1$ was studied earlier in [31] (see also [24]).*

(b) *Let $(x^n, k^n) = SP_{l^n}(y^n)$, $(x, k) = SP_l(y)$. By (2.5) and (2.6) it is clear that if (y^n, l^n) tends to (y, l) in the uniform norm then (x^n, k^n) tends to (x, k) in the uniform norm. From this one can deduce that*

$$(y^n, l^n) \rightarrow (y, l) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \Rightarrow (x^n, k^n, y^n, l^n) \rightarrow (x, k, y, l) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$$

and if $\{(y^n, l^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ then $\{(x^n, k^n, y^n, l^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$ (see e.g. [31]). From Corollary 2.3 it also follows that if (y^n, l^n) tends to (y, l) in the variation norm then (x^n, k^n) tends to (x, k) in the variation norm.

Remark 2.5 Let $(x, k) = SP_l(y)$. Since $k_t = \sup_{s \leq t} (y_s - l_s)^-$, for any $T \in \mathbb{R}^+$ we have

$$\bar{V}_p(k)_T \leq d \sup_{t \leq T} |y_t| + d \sup_{t \leq T} |l_t|.$$

and

$$\bar{V}_p(x)_T \leq (d+1)\bar{V}_p(y)_T + d \sup_{t \leq T} |l_t|.$$

3 Deterministic integral equations

Let $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$, $z \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $V_q(x)_T < \infty$, $V_p(z)_T < \infty$, $T \in \mathbb{R}^+$, where $1/p + 1/q > 1$, $p, q \geq 1$. It is well known (see, e.g., [6, 7, 8, 34]) that the Riemann-Stieltjes integral $\int_0^\cdot x_{s-} dz_s$ is a well defined càdlàg function such that for any $a < b$,

$$V_p\left(\int_a^\cdot x_{s-} dz_s\right)_{[a,b]} \leq C_{p,q} \bar{V}_q(x)_{[a,b]} V_p(z)_{[a,b]}, \quad (3.20)$$

where $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and ζ denotes the Riemann zeta function, i.e. $\zeta(x) = \sum_{n=1}^\infty 1/n^x$.

Let $a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $z, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $V_1(a)_T$, $V_p(z)_T$, $T \in \mathbb{R}^+$, and $x_0 \geq l_0$. We consider equations with constraints of the form

$$x_t = x_0 + \int_0^t f(x_{s-}) da_s + \int_0^t g(x_{s-}) dz_s + k_t, \quad t \in \mathbb{R}^+, \quad (3.21)$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d \rightarrow \mathbb{M}^d$ are given functions and the integral with respect to z is a Riemann-Stieltjes integral.

Definition 3.1 We say that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is a solution of (3.21) if $V_p(x)_T < \infty$, $T \in \mathbb{R}^+$, and $(x, k) = SP_l(y)$, where

$$y_t = x_0 + \int_0^t f(x_{s-}) da_s + \int_0^t g(x_{s-}) dz_s, \quad t \in \mathbb{R}^+.$$

We will need the following conditions.

(H1) (a) $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous and satisfies the linear growth condition, i.e. there is $L > 0$ such that

$$|f(x)| \leq L(1 + |x|), \quad x \in \mathbb{R}^d.$$

(b) $g : \mathbb{R}^d \rightarrow \mathbb{M}^d$ is Hölder continuous function of order $\alpha \in (p-1, 1]$, i.e. there is $C_\alpha > 0$ such that

$$\|g(x) - g(y)\| \leq C_\alpha |x - y|^\alpha, \quad x, y \in \mathbb{R}^d.$$

(H2) (a) $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz continuous, i.e. for any $k \in \mathbb{N}$ there is $L_k > 0$ such that

$$|f(x) - f(y)| \leq L_k |x - y|, \quad |x|, |y| \leq k.$$

(b) $g : \mathbb{R}^d \rightarrow \mathbb{M}^d$, its each component $g_{i,j}$ is differentiable and there are $\gamma \in (p-1, 1]$ and $C_{k,\gamma} > 0$ such that for every $k \in \mathbb{N}$,

$$|\nabla_x g_{i,j}(x) - \nabla_x g_{i,j}(y)| \leq C_{k,\gamma} |x - y|^\gamma, \quad |x|, |y| \leq k, \quad i, j = 1, \dots, d.$$

Similar sets of conditions were considered in papers on equations without constraints driven by functions (processes) with bounded p -variation (see, e.g., [6, 12, 17, 18, 20, 22, 23]).

The outline of the rest of Section 3 is as follows. First we study convergence of solutions of equations of the type (3.21) in the Skorokhod topology J_1 . In our proofs we use a general tightness criterion and a functional limit theorem for sequences of integrals with respect to càdlàg functions (see Appendix). As a simple corollary to our convergence result we show that under (H1) there exists a solution of (3.21). Next, assuming additionally (H2), we prove that (3.21) has a unique solution (x, k) . Under (H1) and (H2), we show at the end of Section 3, that (x, k) can be approximated by simple and easy to implement approximation schemes.

Let $z^n, l^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $a^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ be such that $x_0^n \geq l_0^n$ and $V_1(a^n)_T, V_p(z^n)_T < \infty$, $T \in \mathbb{R}^+$. We will consider solutions $(x^n, k^n) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ of equations with constraints of the form

$$x_t^n = x_0^n + \int_0^t f(x_{s-}^n) da_s^n + \int_0^t g(x_{s-}^n) dz_s^n + k_t^n, \quad t \in \mathbb{R}^+, \quad (3.22)$$

i.e. $(x^n, k^n) = SP_l^n(x_0^n + \int_0^\cdot f(x_{s-}^n) da_s^n + \int_0^\cdot g(x_{s-}^n) dz_s^n)$ and $V_p(x^n)_T < \infty$, $T \in \mathbb{R}^+$.

Theorem 3.2 *Suppose that functions f and g satisfy (H1). Let $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}, \{l^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be sequences such that $\sup_n V_1(a^n)_T < \infty$, $\sup_n V_p(z^n)_T < \infty$, $T \in \mathbb{R}^+$ and*

$$(x_0^n, a^n, z^n, l^n) \longrightarrow (x_0, a, z, l) \text{ in } \mathbb{R}^d \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1}).$$

If $\{(x^n, k^n)\}$ is a sequence of solutions of (3.22) then

$$\{(x^n, k^n)\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$$

and its every limit point is a solution of (3.21).

PROOF. The lemma below will be our main tool in the proof.

Lemma 3.3 *Assume f and g satisfy (H1). Let (x, k) be a solution of (3.21) and let $b, T > 0$. If*

$$\max(V_1(a)_T, V_p(z)_T, \sup_{t \leq T} |l_t|) \leq b$$

then there is $\bar{C} = C(d, p, \alpha, L, g(0), x_0, b) > 0$ such that $\bar{V}_p(x)_T \leq \bar{C}$.

PROOF. By Remark 2.5, for any $t \leq T$,

$$\begin{aligned} \bar{V}_p(x)_t &\leq (d+1)\bar{V}_p(y)_t + d \sup_{s \leq t} |l_s| \\ &\leq (d+1) \left[|x_0| + V_p\left(\int_0^\cdot f(x_{s-}) da_s\right)_t + V_p\left(\int_0^\cdot g(x_{s-}) dz_s\right)_t \right] + d \sup_{s \leq t} |l_s|. \end{aligned}$$

We have

$$V_p\left(\int_0^\cdot f(x_{s-}) da_s\right)_t \leq V_1(a)_t \sup_{s \leq t} |f(x_{s-})| \leq L V_1(a)_t (1 + \bar{V}_p(x)_t)$$

and, by (3.20),

$$\begin{aligned}
V_p\left(\int_0^\cdot g(x_{s-}) dz_s\right)_t &\leq C_{p,p/\alpha} \bar{V}_{p/\alpha}(g(x))_t V_p(z)_t \\
&\leq C_{p,p/\alpha} (C_\alpha V_p^\alpha(x)_t + C_\alpha |x_0|^\alpha + |g(0)|) V_p(z)_t \\
&\leq C_{p,p/\alpha} [C_\alpha (\alpha \bar{V}_p(x)_t + 2(1-\alpha)) + |g(0)|] V_p(z)_t \\
&\leq D V_p(z)_t (1 + \bar{V}_p(x)_t),
\end{aligned}$$

where $D = C_{p,p/\alpha} (C_\alpha (2-\alpha) + |g(0)|)$.

Set $t_1 = \inf\{t; LV_1(a)_t > \frac{1}{4(d+1)} \text{ or } D V_p(z)_t > \frac{1}{4(d+1)}\} \wedge T$. By the above,

$$\bar{V}_p(x)_{[0,t_1]} \leq (d+1)|x_0| + \frac{1}{2}(1 + \bar{V}_p(x)_{[0,t_1]}) + d \sup_{s \leq t_1} |l_s|,$$

which implies that $\bar{V}_p(x)_{[0,t_1]} \leq 2(d+1)|x_0| + 1 + 2d \sup_{s \leq t_1} |l_s|$. Since

$$\begin{aligned}
|\Delta x_{t_1}| &\leq |f(x_{t_1-}) \Delta a_{t_1}| + |g(x_{t_1-}) \Delta z_{t_1}| + |\Delta l_{t_1}| \\
&\leq (L(1 + |x_{t_1-}|) + C_\alpha |x_{t_1-}|^\alpha + |g(0)| + 2)b,
\end{aligned}$$

there exist $C_1, C_2 > 0$ depending only on $d, p, \alpha, L, g(0), b$ such that

$$\bar{V}_p(x)_{[0,t_1]} \leq C_1 + C_2 |x_0|.$$

Set $t_k = \inf\{t > t_{k-1}; LV_1(a)_{[t_{k-1},t]} > \frac{1}{4(d+1)} \text{ or } D V_p(z)_{[t_{k-1},t]} > \frac{1}{4(d+1)}\} \wedge T$, $k = 2, 3, \dots$, and observe that for the same constants C_1, C_2 ,

$$V_p(x)_{[t_{k-1},t_k]} \leq C_1 + C_2 |x_{t_{k-1}}| \leq C_1 + C_2 \bar{V}_p(x)_{[0,t_{k-1}]}.$$

What is left is to show that $m = \inf\{k; t_k = T\}$ is finite and depends only on $p, \alpha, L, g(0), b$. To see this, let us observe that

$$\begin{aligned}
m \left(\frac{1}{4(d+1)} \right)^p &\leq \sum_{k=1}^m LV_1(a)_{[t_{k-1},t_k]} + D^p v_p(z)_{[t_{k-1},t_k]} \\
&\leq Lb + D^p b^p,
\end{aligned}$$

which yields $m \leq (4(d+1))^p [Lb + D^p b^p]$. This completes the proof. \square

By Lemma 3.3 $\sup_n \bar{V}_p(x^n)_T < \infty$. Since $\bar{V}_{p/\alpha}(g(x^n))_T \leq C_\alpha V_p^\alpha(x^n)_T + |g(x_0^n)|$, we also have

$$\sup_n \bar{V}_{p/\alpha}(g(x^n))_T < \infty, \quad T \in \mathbb{R}^+. \quad (3.23)$$

Since f is continuous, there exists a sequence of Lipschitz continuous functions $\{f^k\}$ such that for any compact $K \subset \mathbb{R}^d$, $\sup_{u \in K} |f^k(u) - f(u)| \rightarrow 0$. Hence and from the fact that $\sup_n \bar{V}_p(x^n)_T < \infty$, $T \in \mathbb{R}^+$, it follows that

$$\sup_n \bar{V}_p(f^k(x^n))_T < \infty, \quad T \in \mathbb{R}^+, \quad k \in \mathbb{N} \quad (3.24)$$

and

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{t \leq T} |f^k(x_t^n) - f(x_t)| = 0, \quad T \in \mathbb{R}^+. \quad (3.25)$$

Putting $q = p/\alpha$ in Corollary 5.2 and using (3.23), (3.24) we show that for any $k \in \mathbb{N}$,

$$\left\{ \left(\int_0^\cdot f^k(x_{s-}^n) da_s^n, a^n, \int_0^\cdot g(x_{s-}^n) dz_s^n, z^n \right) \right\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

By the above and (3.25),

$$\left\{ \left(\int_0^\cdot f(x_{s-}^n) da_s^n, a^n, \int_0^\cdot g(x_{s-}^n) dz_s^n, z^n \right) \right\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

Therefore, $\{(a^n, y^n, z^n, l^n)\}$, with $y^n = x_0^n + \int_0^\cdot f(x_{s-}^n) da_s^n + \int_0^\cdot g(x_{s-}^n) dz_s^n$, is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$. Now, put $(x^n, k^n) = SP_{l^n}(y^n)$, $n \in \mathbb{N}$, and observe that by Remark 2.4(b),

$$\{(x^n, a^n, z^n, l^n)\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

Assume that $(x^n, a^n, z^n, l^n) \rightarrow (x, a, z, l)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$. By Corollary 5.4, $(y^n, l^n) \rightarrow (y, l)$, where $y = x_0 + \int_0^\cdot f(x_{s-}) da_s + \int_0^\cdot g(x_{s-}) dz_s$. Consequently, by Remark 2.4(a),

$$(x^n, k^n) = SP_{l^n}(y^n) \rightarrow SP_l(y) = (x, k) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

□

Corollary 3.4 *Assume f, g satisfy (H1) and let $a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $z, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $V_1(a)_T < \infty$, $V_p(z)_T < \infty$ with $x_0 \geq l_0$. Set $x_0^n = x_0$, $k_0^n = 0$ and*

$$\begin{cases} \Delta y_{(k+1)/n}^n &= f(x_{n/k}^n)(a_{(k+1)/n} - a_{k/n}) + g(x_{k/n}^n)(z_{(k+1)/n} - z_{k/n}), \\ x_{(k+1)/n}^n &= \max(x_{k/n}^n + \Delta y_{(k+1)/n}^n, l_{(k+1)/n}), \\ k_{(k+1)/n}^n &= k_k^n + (x_{(k+1)/n}^n - x_{k/n}^n) - \Delta y_{(k+1)/n}^n \end{cases}$$

and $x_t^n = x_{k/n}^n$, $k_t^n = k_{k/n}^n$, $l_t^n = l_{k/n}^n$, $t \in [k/n, (k+1)/n)$, $k \in \mathbb{N} \cup \{0\}$. Then $\{(x^n, k^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ and its every limit point is a solution of (3.21). Consequently, equation (3.21) has a solution (possibly nonunique).

PROOF. It suffices to observe that (x^n, k^n) is a solution of (3.22) with $a_t^n = a_{k/n}$, $z_t^n = z_{k/n}$, $l_t^n = l_{k/n}$, $t \in [k/n, (k+1)/n)$, $k \in \mathbb{N} \cup \{0\}$. Also observe that $\sup_n V_1(a^n)_T \leq V_1(a)_T < \infty$, $\sup_n V_p(z^n)_T \leq V_p(z)_T < \infty$ for $T \in \mathbb{R}^+$ and

$$(a^n, z^n, l^n) \longrightarrow (a, z, l) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1}).$$

Therefore the result follows from Theorem 3.2. □

Theorem 3.5 *Assume f, g satisfy (H1) and (H2). Then there exists a unique solution (x, k) of (3.21).*

PROOF. Assume that there exist two solutions (x^j, k^j) , $j = 1, 2$.

Step 1. We first replace (H2) by stronger condition

(H2*) (a) $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz continuous, i.e. there is $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \quad u \in \mathbb{R}^d.$$

(b) $g : \mathbb{R}^d \rightarrow \mathbb{M}^d$, its each component $g_{i,j}$ is differentiable,

$$C = \max_{i,j} \sup_x |\nabla_x g_{i,j}(x)| < \infty$$

and there are $\gamma \in (p-1, 1]$ and $C_\gamma > 0$ such that

$$|\nabla_x g_{i,j}(x) - \nabla_x g_{i,j}(y)| \leq C_\gamma |x - y|^\gamma, \quad x, y \in \mathbb{R}^d, \quad i, j = 1, \dots, d.$$

Fix $T \in \mathbb{R}^+$. By Corollary 2.3, for any $t \leq T$ we have

$$\begin{aligned} \bar{V}_p(x^1 - x^2)_t &\leq (d+1)\bar{V}_p\left(\int_0^\cdot f(x_{s-}^1) - f(x_{s-}^2) da_s\right)_t \\ &\quad + (d+1)\bar{V}_p\left(\int_0^\cdot g(x_{s-}^1) - g(x_{s-}^2) dz_s\right)_t. \end{aligned}$$

Moreover,

$$\bar{V}_p\left(\int_0^\cdot f(x_{s-}^1) - f(x_{s-}^2) da_s\right)_t \leq LV_1(a)_T \sup_{s \leq t} |x_s^1 - x_s^2| \leq LV_1(a)_t \bar{V}_p(x^1 - x^2)_t$$

and by (3.20),

$$\bar{V}_p\left(\int_0^\cdot g(x_{s-}^1) - g(x_{s-}^2) dy_s\right)_t \leq C_{p,p/\gamma} \bar{V}_{p/\gamma}(g(x^1) - g(x^2))_t V_p(z)_t.$$

By [6, Theorem 2] for $i, j = 1, \dots, d$ we have

$$V_{p/\gamma}(g_{i,j}(x^1) - g_{i,j}(x^2))_t \leq C V_{p/\gamma}(x^1 - x^2)_t + C_\gamma \sup_{s \leq t} |x_s^1 - x_s^2| (V_p(x^1)_T)^\gamma.$$

Therefore

$$\begin{aligned} V_{p/\gamma}(g(x^1) - g(x^2))_t &\leq \sum_{i,j=1}^d V_{p/\gamma}(g_{i,j}(x^1) - g_{i,j}(x^2))_t \\ &\leq \tilde{C}_1 V_{p/\gamma}(x^1 - x^2)_t + \tilde{C}_2 \sup_{s \leq t} |x_s^1 - x_s^2| (V_p(x^1)_T)^\gamma, \end{aligned}$$

where $\tilde{C}_1 = C^{d^2}$, $\tilde{C}_2 = (C_\gamma)^{d^2}$. Set

$$t_1 = \inf\{t; \max[LV_1(a)_t, C_{p,p/\gamma}(\tilde{C}_1 + \tilde{C}_2 V_p(x^1)_t^\gamma) V_p(z)_t] > \frac{1}{4(d+1)}\} \wedge T.$$

Then $\bar{V}_p(x^1 - x^2)_{[0,t_1]} \leq \frac{1}{2} \bar{V}_p(x^1 - x^2)_{[0,t_1]}$, thus $x^1 = x^2$ on $[0, t_1)$. Since for $j = 1, 2$ we have

$$x_{t_1}^j = \max(x_{t_1-}^j + f(x_{t_1-}^j) \Delta a_{t_1} + g(x_{t_1-}^j) \Delta y_{t_1}, l_{t_1}),$$

$x_{t_1}^1 = x_{t_1}^2$, too. For $k \geq 2$ set

$$t_k = \inf\{t > t_{k-1}; \max[LV_1(a)_{[t_{k-1}, t)}, \\ C_{p, p/\gamma}(\tilde{C}_1 + \tilde{C}_2 V_p(x^1)_T^\gamma) V_p(z)_{[t_{k-1}, t)}] > \frac{1}{4(d+1)}\} \wedge T.$$

Arguing as above we show recurrently that $x^1 = x^2$ on each interval $[t_{k-1}, t_k]$. Since by the same arguments as in the proof of Theorem 3.2, $m = \inf\{k; t_k = T\}$ is finite, $x^1 = x^2$ on the interval $[0, T]$, which completes the proof under (H2*).

Step 2. The general case. Set $s_k = \inf\{t; \max(|x_t^1|, |x_t^2|) > k\}$, $k \in \mathbb{N}$. By the first part of the proof,

$$x_t^1 = x_t^2, \quad t < s_k, \quad k \in \mathbb{N}.$$

Since $s_k \rightarrow \infty$, this proves the corollary. \square

Corollary 3.6 *Assume (H1) and (H2). Let $a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ be such that $V_1(a)_T < \infty$, $z, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, with $V_p(z)_T < \infty$, $T \in \mathbb{R}^+$ and $x_0 \geq l_0$. Let $\{(x^n, k^n)\}$ be a sequence of approximations defined in Corollary 3.4. Then*

$$(x^n, k^n) \longrightarrow (x, k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \quad (3.26)$$

and for any $T \in \mathbb{R}^+$,

$$\max_{k/n \leq T} |x_{k/n}^n - x_{k/n}| \longrightarrow 0 \quad \text{and} \quad \max_{k/n \leq T} |k_{k/n}^n - k_{k/n}| \longrightarrow 0, \quad (3.27)$$

where (x, k) is a unique solution of (3.21).

PROOF. The convergence (3.26) easily follows from Corollary 3.4. If we set $x_t^{(n)} = x_{k/n}^n$, $k_t^{(n)} = k_{k/n}^n$, $t \in [k/n, (k+1)/n]$, $k \in \mathbb{N} \cup \{0\}$, then by [14, Proposition 2.2]

$$(x^n, x^{(n)}, k^n, k^{(n)}) \longrightarrow (x, x, k, k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$$

Consequently, $x^n - x^{(n)} \rightarrow 0$ and $k^n - k^{(n)} \rightarrow 0$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, which is equivalent to (3.27). \square

Corollary 3.7 *Assume (H1) and (H2). Let a, z, l satisfy assumptions of Corollary 3.6 and let $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}, \{l^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that for all $T \in \mathbb{R}^+$ $\sup_n V_1(a^n)_T < \infty$, $\sup_n V_p(z^n)_T < \infty$, and*

$$\sup_{t \leq T} (|a_t^n - a_t| + |z_t^n - z_t| + |l_t^n - l_t|) \longrightarrow 0, \quad T \in \mathbb{R}^+. \quad (3.28)$$

If $\{(x^n, k^n)\}$ is a sequence of solutions of (3.22) then

$$\sup_{t \leq T} (|x_t^n - x_t| + |k_t^n - k_t|) \longrightarrow 0, \quad T \in \mathbb{R}^+,$$

where (x, k) is a unique solution of (3.21).

PROOF. By Theorem 3.2,

$$(x^n, k^n, a^n, z^n, l^n) \longrightarrow (x, k, a, z, l) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d+1}).$$

Since for any $t \in \mathbb{R}^+$,

$$\Delta x_t \neq 0 \implies \Delta a_t \neq 0 \text{ or } \Delta z_t \neq 0 \text{ or } \Delta l_t \neq 0,$$

[27, Lemma C] shows that $\sup_{t \leq T} |x_t^n - x_t| \rightarrow 0$, $T \in \mathbb{R}^+$. Similarly we show the uniform convergence of k^n to k . \square

Corollary 3.8 *Assume (H1) and (H2). Let a, z, l satisfy assumptions of Corollary 3.6. Set $x_0^n = x_0$, $k_0^n = 0$, $t_0^n = 0$,*

$$t_k^n = \inf\{t > t_{k-1}^n; \max(|\Delta a_t|, |\Delta z_t|, |\Delta l_t|) > \frac{1}{n}\} \wedge (t_{k-1}^n + \frac{1}{n}), \quad k \in \mathbb{N},$$

and

$$\begin{cases} \Delta y_{t_{k+1}^n}^n &= f(x_{t_k^n}^n)(a_{t_{k+1}^n}^n - a_{t_k^n}^n) + g(x_{t_k^n}^n)(z_{t_{k+1}^n}^n - z_{t_k^n}^n), \\ x_{t_{k+1}^n}^n &= \max(x_{t_k^n}^n + \Delta y_{t_{k+1}^n}^n, l_{t_{k+1}^n}^n), \\ k_{t_{k+1}^n}^n &= k_{t_k^n}^n + (x_{t_{k+1}^n}^n - x_{t_k^n}^n) - \Delta y_{t_{k+1}^n}^n \end{cases}$$

and $x_t^n = x_{t_k^n}^n$, $k_t^n = k_{t_k^n}^n$, $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$. If $\{(x^n, k^n)\}$ is a sequence of solutions of (3.22) then

$$\sup_{t \leq T} (|x_t^n - x_t| + |k_t^n - k_t|) \longrightarrow 0, \quad T \in \mathbb{R}^+,$$

where (x, k) is a unique solution of (3.21).

PROOF. Observe that $\{(x^n, k^n)\}$ is a sequence of solutions of (3.22) with $a_t^n = a_{t_k^n}^n$, $z_t^n = z_{t_k^n}^n$, $l_t^n = l_{t_k^n}^n$, $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$, and that

$$\sup_n V_1(a^n)_T \leq V_1(a)_T < \infty, \quad \sup_n V_p(z^n)_T \leq V_p(z)_T < \infty, \quad T \in \mathbb{R}^+.$$

Moreover, simple calculations show that (3.28) is satisfied. Therefore the desired result follows from Corollary 3.7. \square

4 SDEs with constraints

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and let A be an (\mathcal{F}_t) adapted process with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R})$, Z, L be (\mathcal{F}_t) adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that for any $T \in \mathbb{R}^+$, $P(V_1(A)_T < \infty) = 1$ and $P(V_p(Z)_T < \infty) = 1$. Note that Z need not to be a semimartingale. However, it is a p -semimartingale and a Dirichlet process in the sense considered in [17, 18] and [5].

Definition 4.1 *Let $X_0 \geq L_0$. We say that a pair (X, K) of (\mathcal{F}_t) adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $P(V_p(X)_T < \infty) = 1$ for $T \in \mathbb{R}^+$ is a strong solution of (1.1) if $(X, K) = SP_L(Y)$, where*

$$Y_t = X_0 + \int_0^t f(X_{s-}) dA_s + \int_0^t g(X_{s-}) dZ_s, \quad t \in \mathbb{R}^+.$$

Theorem 4.2 Assume (H1) and (H2). If $X_0 \geq L_0$ then equation (1.1) has a unique strong solution (X, K) . Moreover, if we define (X^n, K^n) as

$$X_t^n = X_{\tau_k^n}^n, \quad K_t^n = K_{\tau_k^n}^n, \quad t \in [\tau_k^n, \tau_{k+1}^n), \quad k \in \mathbb{N} \cup \{0\},$$

where $X_0^n = X_0$, $K_0^n = 0$ and

$$\begin{cases} \Delta Y_{\tau_{k+1}^n}^n &= f(X_{\tau_k^n}^n)(A_{\tau_{k+1}^n} - A_{\tau_k^n}) + g(X_{\tau_k^n}^n)(Z_{\tau_{k+1}^n} - Z_{\tau_k^n}), \\ X_{\tau_{k+1}^n}^n &= \max(X_{\tau_k^n}^n + \Delta Y_{\tau_{k+1}^n}^n, L_{\tau_{k+1}^n}), \\ K_{\tau_{k+1}^n}^n &= K_{\tau_k^n}^n + (X_{\tau_{k+1}^n}^n - X_{\tau_k^n}^n) - \Delta Y_{\tau_{k+1}^n}^n \end{cases}$$

with $\tau_0^n = 0$, $\tau_k^n = \inf\{t > \tau_{k-1}^n; \max(|\Delta A_t|, |\Delta Z_t|, |\Delta L_t|) > \frac{1}{n}\} \wedge (\tau_{k-1}^n + \frac{1}{n})$, $n, k \in \mathbb{N}$, then for any $T \in \mathbb{R}^+$,

$$\sup_{t \leq T} |X_t^n - X_t| \rightarrow 0, \quad P\text{-a.s.}, \quad \sup_{t \leq T} |K_t^n - K_t| \rightarrow 0, \quad P\text{-a.s.},$$

PROOF. From Theorem 3.5 we deduce that for every $\omega \in \Omega$ there exists a unique solution $(X(\omega), K(\omega)) = SP_{L(\omega)}(Y(\omega))$. Moreover, by Corollary 3.8, for every $\omega \in \Omega$ and $T \in \mathbb{R}^+$,

$$\sup_{t \leq T} |X_t^n(\omega) - X_t(\omega)| \rightarrow 0, \quad \sup_{t \leq T} |K_t^n(\omega) - K_t(\omega)| \rightarrow 0.$$

Since for any $n \in \mathbb{N}$ the pair (X^n, K^n) is (\mathcal{F}_t) adapted, the pair of limit processes (X, K) is (\mathcal{F}_t) adapted as well, which completes the proof. \square

Corollary 4.3 Under the assumptions of Theorem 4.2 with random sequences of partitions $\{\tau_n^k\}$ replaced by constant sequences $\{\frac{k}{n}\}$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, we have

$$(X^n, K^n) \longrightarrow (X, K) \quad P\text{-a.s. in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

where (X, K) is a unique strong solution of (1.1).

PROOF. It suffices to apply Corollary 3.6. \square

Let B^H be a fractional Brownian motion (fBm) with Hurst index $H > 1/2$, i.e. a continuous centered Gaussian process with covariance

$$EB_{t_2}^H B_{t_1}^H = \frac{1}{2}(t_2^{2H} + t_1^{2H} - |t_2 - t_1|^{2H}), \quad t_1, t_2 \in \mathbb{R}^+.$$

Let $Z^H = \int_0^\cdot \sigma_s dB_s^H$, where $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a measurable function such that $\|\sigma\|_{\mathbb{L}_{[0,T]}^{1/H}} := (\int_0^T |\sigma_s|^{1/H} ds)^H < \infty$, $T \in \mathbb{R}^+$. Then Z^H is also a continuous centered Gaussian process with continuous trajectories. Moreover, if $p > 1/H$ then

$$P(V_p(Z^H)_T < \infty) = 1, \quad T \in \mathbb{R}^+ \quad (4.29)$$

(see, e.g., [12, Proposition 2.1]). Note also that Z^H is a Dirichlet process from the class $\mathcal{D}^{1/H}$ studied in [4].

We now show how to apply our results to fractional SDEs with constraints of the form (1.4). Let $B^H = (B^{H,1}, \dots, B^{H,d})$, where $B^{H,1}, \dots, B^{H,d}$ are independent fractional Brownian motions, and let $Z^H = (Z^{H,1}, \dots, Z^{H,d})$, where $Z^{H,i} = \int_0^\cdot \sigma_s^i dB_s^{H,i}$ with $\sigma^i : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\|\sigma^i\|_{\mathbb{L}_{[0,T]}^{1/H}} < \infty$, $T > 0$, $i = 1, \dots, d$.

Corollary 4.4 *Assume (H1) and (H2). If $X_0 \geq L_0$ then equation (1.4) has a unique strong solution (X, K) . Moreover, if*

$$X_t^n = X_{k/n}^n, \quad K_t^n = K_{k/n}^n, \quad t \in [k/n, (k+1)/n), \quad k \in \mathbb{N} \cup \{0\}, \quad n \in \mathbb{N},$$

where $X_0^n = X_0$, $K_0^n = 0$ and

$$\begin{cases} \Delta Y_{(k+1)/n}^n &= f(X_{k/n}^n)(a_{(k+1)/n} - a_{k/n}) + g(X_{k/n}^n)(Z_{(k+1)/n}^H - Z_{k/n}^H), \\ X_{(k+1)/n}^n &= \max(X_{k/n}^n + \Delta Y_{(k+1)/n}^n, L_{(k+1)/n}), \\ K_{(k+1)/n}^n &= K_{k/n}^n + (X_{(k+1)/n}^n - X_{k/n}^n) - \Delta Y_{(k+1)/n}^n, \end{cases}$$

then for any $T \in \mathbb{R}^+$,

$$\sup_{t \leq T} |X_t^n - X_t| \rightarrow 0, \quad P\text{-a.s.}, \quad \sup_{t \leq T} |K_t^n - K_t| \rightarrow 0, \quad P\text{-a.s.},$$

PROOF. It suffices to apply Corollary 4.3 and use the facts that a is a continuous function and Z^H has continuous trajectories. \square

Remark 4.5 *To approximate Z^H one can use the methods developed in [12, 32].*

5 Appendix

Proposition 5.1 *Let $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$, $\{z^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and*

$$\sup_n \bar{V}_q(x^n)_T < \infty, \quad \sup_n V_p(z^n)_T < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geq 1$. If $\{z^n\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ then

$$\{(\int_0^\cdot x_{s-}^n dz_s^n, z^n)\} \quad \text{is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

PROOF. Without loss of generality we may and will assume that

$$z^n \longrightarrow z \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).$$

We follow arguments from the proof of [21, Proposition 3.3] and [27, Proposition 2]. Let $t_{k,0}^n = 0$, $t_{k,i+1}^n = \min(t_{k,i}^n + \delta_{k,i}, \inf\{t > t_{k,i}^n, |\Delta z_t^n| > \delta_k\})$ and $t_{k,0} = 0$, $t_{k,i+1} = \min(t_{k,i} + \delta_{k,i}, \inf\{t > t_{k,i}, |\Delta z_t| > \delta_k\})$, where $\{\delta_k\}$, $\{\{\delta_{k,i}\}\}$ are families of constants such that $\delta_k \downarrow 0$, $|\Delta z_t| \neq \delta_k, t \in \mathbb{R}^+$, $\delta_k/2 \leq \delta_{k,i} \leq \delta_k$ and $|\Delta z_{t_{k,i} + \delta_{k,i}}| = 0$, $i \in \mathbb{N} \cup \{0\}$, $k, n \in \mathbb{N}$. Define

$$z_t^{n,(k)} = z_{t_{k,i}^n}^n, \quad t \in [t_{k,i}^n, t_{k,i+1}^n) \quad \text{and} \quad z_t^{(k)} = z_{t_{k,i}}, \quad t \in [t_{k,i}, t_{k,i+1})$$

for $i \in \mathbb{N} \cup \{0\}$, $n, k \in \mathbb{N}$. Then $V_p(z^{n,(k)})_T \leq V_p(z^n)_T$, $n \in \mathbb{N}$, $V_p(z^{(k)})_T \leq V_p(z)_T$ for $T \in \mathbb{R}^+$ and

$$t_{k,i}^n \rightarrow t_{k,i}, \quad z_{t_{k,i}^n}^n \rightarrow z_{t_{k,i}}, \quad i \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}. \quad (5.30)$$

Consequently,

$$\sup_n V_p(z^n - z^{n,(k)})_T < \infty, \quad k \in \mathbb{N}. \quad (5.31)$$

and

$$(z^{n,(k)}, z^n) \longrightarrow (z^{(k)}, z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}), \quad k \in \mathbb{N}. \quad (5.32)$$

Moreover,

$$\sup_{t \leq T} |z_t^{(k)} - z_t| \longrightarrow 0, \quad T \in \mathbb{R}^+, \quad (5.33)$$

which together with (5.32) implies that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{t \leq T} |z_t^{n,(k)} - z_t^n| = 0, \quad T \in \mathbb{R}^+. \quad (5.34)$$

On the other hand, $\int_0^t x_{s-}^n dz_s^{n,(k)} = \sum_{j \leq i} x_{t_{k,j}^n}^n (z_{t_{k,j}^n}^n - z_{t_{k,j-1}^n}^n)$, $t \in [t_{k,i}^n, t_{k,i+1}^n)$. Using (5.30), (5.32) and the fact that $\sup_n \bar{V}_q(x^n)_T < \infty$ implies that $\{\sup_{t \leq T} |x_t^n|\}$ is bounded, we conclude that for any $k \in \mathbb{N}$,

$$\{(\int_0^\cdot x_{s-}^n dz_s^{n,(k)}, z^n)\} \quad \text{is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}). \quad (5.35)$$

Let $p' > p$ be such that $1/p' + 1/q > 1$. By (3.20)

$$\sup_{t \leq T} \left| \int_0^t x_{s-}^n d(z^n - z^{n,(k)})_s \right| \leq C_{p',q} \bar{V}_q(x^n)_T V_{p'}(z^n - z^{n,(k)})_T.$$

Moreover

$$V_{p'}(z^n - z^{n,(k)})_T \leq \text{Osc}(z^n - z^{n,(k)})_T^{1-p/p'} V_p(z^n - z^{n,(k)})_T^{p/p'},$$

where $\text{Osc}(x)_T = \sup_{s,t \leq T} |x_t - x_s|$. Since

$$\text{Osc}(z^n - z^{n,(k)})_T \leq 2 \sup_{t \leq T} |z^n - z_t^{n,(k)}| \longrightarrow 0,$$

we deduce from the above that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{t \leq T} \left| \int_0^t x_{s-}^n d(z^n - z^{n,(k)})_s \right| = 0, \quad T \in \mathbb{R}^+. \quad (5.36)$$

Combining (5.35) with (5.36) we get the desired result. \square

Corollary 5.2 *Let $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}, \{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$ be sequences of functions such that*

$$\sup_n \max(V_1(a^n)_T, V_p(z^n)_T, \bar{V}_q(y^n)_T, \bar{V}_q(x^n)_T) < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geq 1$. If

$$\{(a^n, z^n)\} \quad \text{is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d+1})$$

then

$$\{(\int_0^\cdot y_{s-}^n da_s^n, a^n, \int_0^\cdot x_{s-}^n dz_s^n, z^n)\} \quad \text{is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

PROOF. Set $\bar{d} = 2d$ and for every $n \in \mathbb{N}$ define $\bar{z}^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{\bar{d}})$ and $\bar{x}^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{\bar{d}^2})$ by the formulas

$$\bar{z}^{n,i} = \begin{cases} a^n, & i = 1, \dots, d, \\ z^{n,i-d}, & i = d+1, \dots, 2d \end{cases}$$

and

$$(\bar{x}^n)_{i,j} = \begin{cases} y^{n,i}, & i = j = 1, \dots, d, \\ (x^n)_{i-d,j-d}, & i = d+1, \dots, 2d, j = d+1, \dots, 2d, \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 5.1,

$$\{(\int_0^\cdot \bar{x}_{s-}^n d\bar{z}_s^n, \bar{z}^n)\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2\bar{d}}),$$

from which one can deduce the corollary. \square

Proposition 5.3 *Let $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$, $\{z^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and*

$$\sup_n \bar{V}_q(x^n)_T < \infty, \quad \sup_n V_p(z^n)_T < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geq 1$. If $(x^n, z^n) \longrightarrow (x, z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d^2+d})$ then

$$(x^n, z^n, \int_0^\cdot x_{s-}^n dz_s^n) \longrightarrow (x, z, \int_0^\cdot x_{s-} dz_s) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d^2+2d}).$$

PROOF. Since $V_q(x)_T \leq \liminf_{n \rightarrow \infty} V_q(x^n)_T < \infty$ and $V_p(z)_T \leq \liminf_{n \rightarrow \infty} V_p(z^n)_T < \infty$ for $T \in \mathbb{R}^+$, the integral $\int_0^\cdot x_{s-} dz_s$ is well defined. Let $\{\{z^{n,(k)}\}\}$, $\{z^{(k)}\}$ be families of functions defined in the proof of Proposition 5.1. It follows from the equality $\int_0^t x_{s-} dz_s^{(k)} = \sum_{j \leq i} x_{t_{k,j}-}(z_{t_{k,j}} - z_{t_{k,j-1}})$, $t \in [t_{k,i}, t_{k,i+1})$, and (5.30), (5.32) that for any $k \in \mathbb{N}$,

$$(x^n, z^n, \int_0^\cdot x_{s-}^n dz_s^{n,(k)}) \longrightarrow (x, z, \int_0^\cdot x_{s-} dz_s^{(k)}) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d^2+2d}). \quad (5.37)$$

As in the proof of (5.36) we check that $\lim_{k \rightarrow \infty} \sup_{t \leq T} |\int_0^t x_{s-} d(z - z^{(k)})_s| = 0$, $T \in \mathbb{R}^+$. From this and (5.36), (5.37) the result follows. \square Using arguments from the proof of Corollary 5.2 it is easy to check that Proposition 5.3 implies the following corollary.

Corollary 5.4 *Let $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}, \{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$ be sequences of functions such that*

$$\sup_n \max(V_1(a^n)_T, V_p(z^n)_T, \bar{V}_q(y^n)_T, \bar{V}_q(x^n)_T) < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geq 1$. If

$$(y^n, a^n, x^n, z^n) \longrightarrow (y, a, x, z) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d^2+2d+1})$$

then

$$(\int_0^\cdot y_{s-}^n da_s^n, a^n, \int_0^\cdot x_{s-}^n dz_s^n, z^n) \longrightarrow (\int_0^\cdot y_{s-} da_s, a, \int_0^\cdot x_{s-} dz_s, z)$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$.

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